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Feedback Stabilization of Inverted Pendulum Models

Bruce Cox
Virginia Commonwealth University

College of Humanities and Science
Virginia Commonwealth University

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William Terrell, Ph.D – Department of Mathematics

Jason Merrick, Ph.D – Department of Operations Research

Alison Baski, Ph.D – Department of Physics

John Berglund, Ph.D – Director Graduate Studies for Department of Mathematics

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Robert Holsworth, Ph.D – Dean of the College of Humanities and Science

---

Dr. F. Douglas Boudinot, Dean of the School of Graduate Studies

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By

Bruce A Cox,

B.S. Worcester Polytechnic Institute 1999

Director: William Terrell

Associate Professor, Applied Mathematics

Virginia Commonwealth University

Richmond, Virginia

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Abstract

Many mechanical systems exhibit nonlinear movement and are subject to perturbations from a desired equilibrium state. These perturbations can greatly reduce the efficiency of the systems. It is therefore desirous to analyze the asymptotic stabilizability of an equilibrium solution of nonlinear systems; an excellent method of performing these analyses is through study of Jacobian linearization's and their properties.

Two enlightening examples of nonlinear mechanical systems are the Simple Inverted Pendulum and the Inverted Pendulum on a Cart (PoC). These examples provide insight into both the feasibility and usability of Jacobian linearizations of nonlinear systems, as well as demonstrate the concepts of local stability, observability, controllability and detectability of linearized systems under varying parameters. Some examples of constant disturbances and effects are considered. The ultimate goal is to examine stabilizability, through both static and dynamic feedback controllers, of mechanical systems

Chapter 1: Introduction

Section 1.1: What is Feedback Control? Why Study it?

Any system, be it of mechanical, electrical or biological origins, is often subject to perturbations from a desired normal state. Feedback control utilizes the current state of the system to determine an appropriate feedback to return the system to this desired state, i.e. to stabilize the system. Compensating for unwanted perturbations is necessary to ensure proper functioning of a system; in many cases the margin of error for a system is very small.

Perturbations can cause severe problems with a sensitive system. In a radio antenna, for example, a few degrees off axis can result in the antenna pointing at the wrong section of sky, resulting in a complete lack of functionality. Other mechanical systems such as bipedal robots and rockets also require feedback control in order to ensure optimal efficiency. These mechanical systems all utilize the same fundamental control principles and techniques. In fact, bipedal locomotion and rockets are directly modeled off of inverted pendulum models.

Feedback control methodologies are used to determine the current state of a mechanical system and induce a mechanical feedback that will return it to the desired state. This paper will be dealing solely with feedback control and stability within mechanical systems.

The State Space representation of a mechanical system is a methodology of writing the system equations.

Definition 1.1:
The ***State Space Representation*** *of a linear time invariant system is:*

(1.1)
$$\frac{dx(t)}{dt} = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

In the future we will write $\dot{x} = \frac{dx}{dt}$. In equations (1.1):

- *x(t)* is the system state at time *t*. In mechanical systems this usually refers to quantities like position and velocity. This is where the name State Space originates.
- *A* is an *n x n* matrix that determines the system dynamics in the absence of any inputs (*u=0*)
- *B* is an *n x m* matrix that determines the interaction between the inputs, *u(t)*, and the system state *x(t).*
- *C* is a *p x n* matrix that determines the interaction between *x(t),* the state space variables and *y(t)* the observable output.

Section 1.2: The Simple Inverted Pendulum and its Linearization

Our first example is a simple inverted pendulum. Imagine a rod with one end fixed in position, yet able to rotate in the vertical plane, so that the rod can swing freely 360 degrees. The other end of the rod has a weight of mass *m* attached to it.

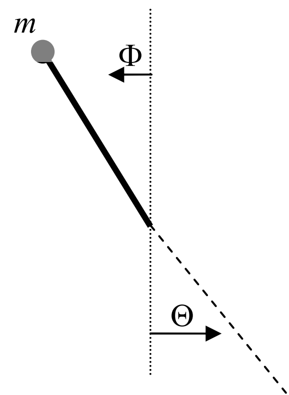

Figure 1.1: The Simple Inverted Pendulum

This pendulum would naturally move to a stable equilibrium with the mass *m* resting at the bottom. Any deviation from this stable equilibrium point could be measured by the angle Θ.

We can also see that an upper vertical equilibrium point exists at $\Theta = \pi$ and $\dot{\Theta} = 0$. Any deviations from this equilibrium are measured by the angle Φ. We define $\Theta = \pi + \Phi$. Logically if the system starts at this vertical equilibrium any small change in initial condition will result in unstable motion. We will prove this mathematically later.

This system equation is:

(1.2) $ml^2\,\ddot{\Theta}(t) + m\,gl\,sin\Theta(t) = u(t)$

If we choose units such that m = 1, l = 1 and g = 1, equation (1.2) then reduces to:

(1.3) $\ddot{\Theta}(t) + sin\,\Theta(t) = u(t)$

Note that in equation (1.3) the only nonlinear term is $sin\,\Theta(t)$. If we can linearize this term, by taking a Taylor expansion and then dropping the higher order terms we end up with a linearized system. However we are fundamentally interested in the stabilization of the nonlinear system around the equilibrium. At first blush this would seem to pose a problem, however it is important to remember that in linearizing the system all we do is drop the higher order terms (H.O.T.s) from the nonlinear part of the system. Since these H.O.T.s are dominated by the lower order terms within the immediate neighborhood of the equilibrium then within that small vicinity the behavior of the nonlinear system can be approximated by the behavior of the linearized system. Furthermore within that immediate neighborhood any feedback control that stabilizes the linear system will also stabilize the nonlinear system. Therefore it is acceptable to examine the linearized system to make calculations easier; this will be laid out in more mathematical detail later.

The first step to linearizing the system is to note that we defined $\Theta = \pi + \Phi$. Thus we can use a trigonometric identity to write:

(1.4) $sin(\Theta) = sin(\pi + \Phi) = sin(\pi)\,cos(\Phi) + sin(\Phi)cos(\pi) = -\,sin(\Phi)$

We then perform a Taylor series expansion on $sin(\Phi)$ keeping only the linear terms.

(1.5) $sin(\Phi) = \Phi + H.O.T.s$

Therefore $sin(\Theta) \approx -\Phi$ for $\Theta - \pi = 0 + \Delta$, where Δ is defined as an arbitrarily small deviation. Then the linearized equation (1.3), about the upper equilibrium, is:

(1.6) $\ddot{\Phi} - \Phi = u(t)$

To find the State Space representation of this linearized system set $x_1 = \Phi$ and $x_2 = \dot{\Phi}$. Then solve for $\dot{x}_1$ and $\dot{x}_2$, in terms of x_1 and x_2. We already know $\dot{x}_1$ explicitly, so all we have to solve for is $\dot{x}_2$.

(1.7) $\dot{x}_2 = x_1 + u(t)$

Translating this into a State Space representation we get:

(1.8) $$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} c_1 & c_2 \end{bmatrix} x$$

Section 1.3: The Inverted Pendulum on a Cart and its Linearization

Consider the Inverted Pendulum on a Cart, as shown by Aplevich [1, page 21]; an inverted pendulum as per Section 1.2, but transpose it to residing on top of a cart of mass *M*. This cart can move forward or backward in only one line, along what we will define as the *x* axis. It is desired to exert a force *u(t)* on the wheeled cart to balance the mass *m*, at the end of pendulum, vertically. Note that *M* >> *m*. See Figure 2.1.

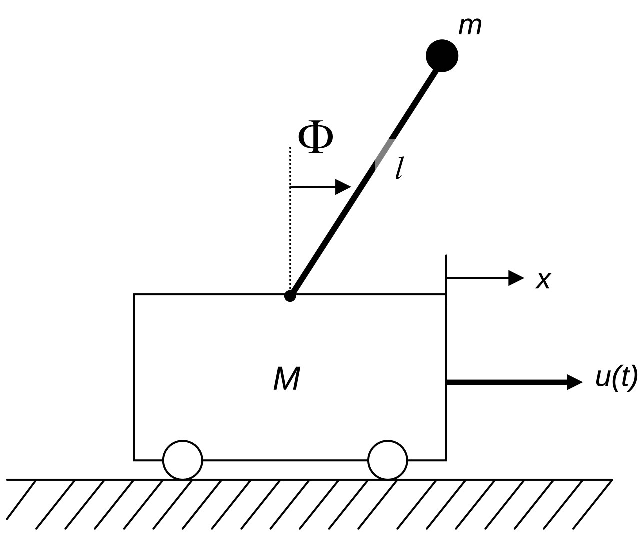

Figure 2.1: Pendulum on a Cart

Ignoring friction and summing up torques at the pendulum pivot we get.

(1.9) $ml^2\ddot{\Phi} - mgl\,sin(\Phi) + ml\ddot{x}\,cos(\Phi) = 0$

The sum of the forces in the horizontal direction is:

(1.10) $M\ddot{x} + m\frac{d^2}{dt^2}\left(x + l\,sin(\Phi)\right) = M\ddot{x} + m\ddot{x} - ml\,sin(\Phi)\dot{\Phi}^2 + ml\,cos(\Phi)\ddot{\Phi} = u(t)$

Since *u(t)* is the force acting on the cart and is the only applied force to the system.

As before we are interested in linearizing the system, so as to use the linearized system to gain insight into the full nonlinear system; there are at least two methodologies that can be used to linearize the system. We shall start by demonstrating the same methodology used in the simple inverted pendulum, wherein we first linearize the nonlinear system equations through Taylor expansions and then we solve for the state space variables.

The Taylor expansion of equations (1.9) and (1.10) can be handled in the same fashion as the Taylor expansion of equation (1.3) from Section 1.2. This results in us dropping the terms with $\Phi^2, \dot{\Phi}^2, \Phi\dot{\Phi}$ and $\Phi\ddot{\Phi}$ while recognizing that $1 + sin\,\Phi \approx 1$, $sin\,\Phi \approx \Phi$ and $cos\,\Phi \approx 1$. Then equations (1.9) and (1.10) can be respectively written linearized as:

(1.11) $ml\ddot{x} + ml^2\ddot{\Phi} - mgl\Phi = 0$

(1.12) $M\ddot{x} + m\ddot{x} + ml\ddot{\Phi} = u(t)$

We now choose the state space variables to be: $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \Phi$, and $x_4 = \dot{\Phi}$. We can now solve equations (1.11) and (1.12) explicitly for $\dot{x}_1, \dot{x}_2, \dot{x}_3$ and $\dot{x}_4$. We already know $\dot{x}_1$ and $\dot{x}_3$, thus we only need to solve for $\dot{x}_2$ and $\dot{x}_4$, which we can do since we have two equations and two unknowns. Then:

(1.13)
$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{-mgx_3 - u}{M} = \frac{-mg}{M}x_3 + \frac{u}{M} \\
\dot{x}_3 &= \dot{x}_4 \\
\dot{x}_4 &= \frac{Mgx_3 + mgx_3 - u}{Ml} = \frac{(M+m)g}{Ml}x_3 - \frac{u}{Ml}
\end{aligned}
$$

Thus the state space representation of the inverted pendulum on a cart is:

(1.14)
$$
\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{-1}{Ml} \end{bmatrix} u
$$

A second more mathematically stringent methodology is also available in order to linearize the system. This second method involves finding the Jacobian Matrix of the system evaluated at the equilibrium.

Definition 1.2:

*The **Jacobian Matrix** is defined as*: $\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$

Choosing the state variables the same as we did for the previous methodology and solving Equations (1.9) and (1.10) for $\ddot{x} = \dot{x}_2, \ddot{\Phi} = \dot{x}_4$ we find the following four time-invariant state equations:

$$\textbf{(1.15)} \quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix}$$

$$\textbf{(1.16)} \quad \begin{bmatrix} \dot{x}_2 \\ \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} f_2 \\ \\ f_4 \end{bmatrix} = \begin{bmatrix} \dfrac{-ml\,sin(x_3)x_4{}^2 + m\,cos(x_3)g\,sin(x_3) - u}{M + m\,sin(x_3)^2} \\ \\ \dfrac{-cos(x_3)ml\,sin(x_3)x_4{}^2 + mg\,sin(x_3) - cos(x_3)u + g\,sin(x_3)M}{(M + m\,sin(x_3)^2)l} \end{bmatrix}$$

Taking a Jacobian Matrix of the system defined by equations (1.15) and (1.16) and evaluating the matrix at the equilibrium, at which all state variables and *u* are zero, then the partial derivatives all evaluate to zero except for the following.

(1.17) $\left.\frac{\partial f_1}{\partial x_2}\right|_0 = 1$

(1.18) $\left.\frac{\partial f_2}{\partial x_3}\right|_0 = {-mg}/{M}$

(1.19) $\left.\frac{\partial f_2}{\partial u}\right|_0 = 1/M$

(1.20) $\left.\frac{\partial f_3}{\partial x_4}\right|_0 = 1$

(1.21) $\left.\frac{\partial f_4}{\partial x_3}\right|_0 = {g(m+M)}/{Ml}$

(1.22) $\left.\frac{\partial f_4}{\partial u}\right|_0 = -1/lM$

Thus the linearization is:

(1.23) $$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{-1}{Ml} \end{bmatrix} u$$

Note that either method of linearizing the Inverted Pendulum on a Cart (PoC) provides identical results.

Chapter 2: Required Background Concepts

Section 2.1: The Null Space of a Matrix

We begin immediately by providing the definition of the Null Space of a matrix.

Definition 2.1:
*The **null space** of an m-by-n matrix A is defined as :* $N(A)=\{x\in\mathbf{R}^{\mathbf{n}}\,|\,Ax=0\}$.

This concept is most easily examined in an example.

Example 2-1:

(2.1) $$A=\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

It is easy to see that the only vector space that satisfies the definition of a null space for the above matrix is:

(2.2) $$N(A) = \left\{ x = \begin{pmatrix} c_o \\ 0 \\ c_1 \\ 0 \end{pmatrix} \right\}$$

This can be seen by multiplying the Matrix *A* by the vector *x* to get.

(2.3) $$Ax = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

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It should be noted that the null space of a matrix equaling the zero subspace, {0}; is equivalent to linear independence of the columns in the matrix.

Section 2.2: Observability

Ideally in any mechanical system we would want to be all-knowing and all-seeing. If we could observe all the aspects of the system, the position, velocity and acceleration of each part, then we could make feedback decisions more easily. Unfortunately, in the real world we are forced to deal with limited resources. Each object we choose to monitor and each fashion in which we choose to monitor it costs money. While developing a mechanical system, we are forced to make choices regarding what we want to observe. However, risk accompanies these choices. If we choose to observe the wrong aspects of the mechanical system, we may not have access to the information we need to make decisions.

Definition 2.2:
We define a system to be ***observable****, over a specified interval: If given the input $u(t)$ and the output $y(t)$ over this time interval, one can uniquely determine the state trajectory $x(t)$ on this interval***.**

This is equivalent to reconstructing $x(0)$ based only on knowledge of $y(t)$ and $u(t)$, since knowledge of $x(0)$ uniquely determines $x(t)$.

Definition 2.3:
We define the Observability Matrix, $\mathbf{W_o}$ *to be:*

(2.4) $$W_0 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Where n is the rank of the system.

Theorem 2.1 [2, page 43]:
A system is observable if and only if W_o *has full column rank.*

Example 2-2:
Let us look at our simple inverted pendulum system from section 1.2. In this system we have two natural choices; we can observe position or we can observe velocity. Let's look at the system with a position measurement to start with. In that case $c_1 = 1$ and $c_2 = 0$. The first step to determine if the system is Observable with only a position measurement is to differentiate $y = [1 \quad 0]x$ to get:

(2.5) $$\dot{y} = \dot{x}_1 = x_2$$

Thus we have a matrix:

(2.6) $$\begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$$

From this matrix we can determine x_1 and x_2 on $[0, t]$ we would therefore say that the system is Observable.

Note that $\begin{bmatrix} y \\ \\ \dot{y} \end{bmatrix} = \boldsymbol{W_o} \begin{bmatrix} x_0 \\ \\ x_1 \end{bmatrix}$

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Example 2-3:
Let's check the observability of the system with only a velocity measurement, I.e. $y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$.

Then:

(2.7) $W_0 = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Thus the system is also observable with only the velocity measurement. This is expected since velocity and position feed information so easily back and forth.

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Definition 2.4:
The **unobservable subspace** is the nullspace of W_o.

Section 2.3: Stability

Another issue of concern prior to dealing directly with feedback control is that of system stability. That is, if a system is perturbed slightly from equilibrium will the system return back to its equilibrium state or will it experience unstable motion. This concept is essential to the idea of feedback control. If a system is naturally stable then there is often no need to impose an artificial feedback control to it, unless one desires to more quickly stabilize the system.

Consider the following system:

(2.8) $\dot{x} = f(x)$ where $f(0) = 0$

Definition 2.5:
The equilibrium point $x = 0$ of (2.8) *is:*

- ***stable*** *if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that*

  $\| x(0) \| < \delta \Rightarrow \| x(t) \| < \varepsilon, \ \forall t \geq 0$

- ***unstable*** *if not stable.*
- ***asymptotically stable*** *if it is stable and δ can be chosen such that*

  $\| x(0) \| < \delta \Rightarrow \lim_{t \to 0} x(t) = 0$

Theorem 2.2 [2, page 20]:
A system, $\dot{x} = Ax$ is (asymptotically) stable if and only if all the eigenvalues, λ_i, of A have negative real part, that is $Re(\lambda_i)$ are negative.

Both our systems from sections 1.2 and 1.3 are nonlinear. We stated in section 1.2 that the stability results for the linearized system would also hold true for the nonlinear system, thus allowing us to study the linearized system for ease of calculation. The following theorem provides the mathematical details behind that ascertation.

Theorem 2.3 [4, page 139]:
Let x = 0 be an equilibrium point for the nonlinear system:

$$\dot{x} = f(x)$$

Where f: D⟶ R^n is continuously differentiable and D is a neighborhood of the origin. Let

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=0}$$

Then,

1. *The origin is asymptotically stable if* $\mathrm{Re}(\lambda_i) < 0$ *for all eigenvalues,* λ_i, *of A.*
2. *The origin is unstable if* $\mathrm{Re}(\lambda_i) > 0$ *for one or more eigenvalues,* λ_i, *of A.*

Example 2-4:
For an example of stability let us look at our simple inverted pendulum system from section 1.2. Here the *A* matrix equals:

(2.9) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

The eigenvalues of this system are λ=1 and λ=-1, therefore as expected the system is unstable around the upper vertical equilibrium. Hence if we desire to asymptotically stabilize the system around the upper equilibrium we need to impose an appropriate feedback control to the system.

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Section 2.4: Controllability

The first step when dealing with systems that are naturally unstable is to determine if they are controllable. "A system is said to be controllable at time t_0 if it is possible by means of an unconstrained control vector to transfer the system from any initial state $x(t_0)$ to any other state in a finite interval of time."[1]

Definition 2.6:
We define a system {A, B} to be **controllable** *over* $[t_0, t_1]$ *(with* $t_1 > t_0$*) if for every pair of states* x_0, x_1 *in* $\mathbf{X}$*, there is a control u such that the solution x of*

(2.10) $\dot{x}(t) = Ax(t) + Bu(t)$ with $x(t_0) = x_0$ *satisfies* $x(t_1) = x_1$.

There are two tests for controllability that shall be used in this paper, the Kalman Controllability Rank Condition and the PBH Controllability Test.

Theorem 2.4 [2, page 57]
The Kalman controllability rank test says that the system $\dot{x} = Ax + Bu$ *is controllable if and only if rank* $W_c := \text{rank}[B \;\; AB \;\; A^2B \;\; \ldots \;\; A^{n-1}B] = n$*, where A is an n x n matrix.*

This is equivalent to saying that the controllability matrix W_c has full row rank.

[1] Katsuhiko Ogata, *Modern Control Engineering*, 4th Ed. New Jersey: Prentice Hall 2002, pp 789

Example 2-5:

Using the Kalman Controllability Rank test on the simple inverted pendulum from section 1.2 we see that:

(2.11) $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(2.12) $AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(2.13) Thus $W_c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

This as we can see has full row rank. We thus determine using the Kalman Controllability Rank test that the inverted pendulum system is controllable.

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Theorem 2.5 [2, page 59]:

The PBH controllability test says that the system $\dot{x} = Ax + Bu$ *is controllable, that is the matrix pair {A, B} is controllable, if and only if* $rank[A - \lambda I \;\; B] = n$ *for all eigenvalues* λ *of A.*

Example 2-6:

Let us also try this test on the simple inverted pendulum. The first step is to find the eigenvalues, we already know the system has the eigenvalues λ=1 and λ=- 1.

For λ=1:

(2.14) $\mathrm{rank}[A - \lambda I \;\; B] = n = rank\begin{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix}\end{bmatrix} = rank\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = 2$

For λ=-1:

(2.15) $\text{rank}[A-\lambda I \;\; B] = n = rank\left[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \;\; \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] = rank\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 2$

Thus as expected the PBH test also shows that the inverted pendulum system is controllable.

Section 2.5: Stabilizability

Interest in controllability is not simply an academic question of whether or not a system is controllable. The fundamental goal of controllability is to asymptotically stabilize a system; if a system is controllable then it is also stabilizable.

Definition 2.7:
*A system is **stabilizable**, by linear feedback, if there exists an m x n matrix **k** such that the system obtained by setting u = kx is asymptotically stable.*

That is, if the system $\dot{x} = (A + Bk)x$ is asymptotically stable per definition 2.5. In the above definition the matrix *k* is defined as a 'gain'. The system $\dot{x} = (A + Bk)x$ is called a ***closed-loop*** control system; a closed-loop system is simply a system that utilizes feedback.

Theorem 2.6 [2, page 123]
The PBH test for linear stabilizability says that the pair {A, B} is stabilizable if and only if $Rank[A - \lambda I \quad B] = n$ *for every eigenvalue* λ *of A with non-negative real part (and thus for every complex* λ *in the right half plane).*

A positive result, i.e. a result where the rank does equal *n*, of the PBH test for linear stability means that all unstable eigenspaces of *A* lie within the controllable subspace. This makes good intuitive sense, once it is realized that the PBH test for linear stabilizability is a direct offshoot of the PBH test for controllability. If there exists an unstable eigenspace for which the PBH test is positive then that eigenspace is controllable and if the eigenspace is controllable then it is stabilizable.

Example 2-7:
For example look at the simple inverted pendulum from section 1.2. There is only one non-negative eigenvector, $\lambda=1$.

For $\lambda=1$:

(2.16) $$\mathrm{rank}[A-\lambda I \;\; B]=n=rank\left[\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}-1\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \;\; \begin{bmatrix}0\\ 1\end{bmatrix}\right]=rank\begin{bmatrix}-1 & 1 & 0\\ 1 & -1 & 1\end{bmatrix}=2$$

Thus the unstable eigenspace associated with $\lambda=1$ is controllable and is thus stabilizable.

Δ

In sections 1.2 and 1.3 we stated that linearizations of nonlinear systems allow us to study important aspects of the linear system and draw conclusions that hold true for the nonlinear system, in a localized area. In section 2.3 we showed that, within a localized area, the stability properties of a nonlinear system can be determined by examining the linearized system. The following theorem provides the basis for determining the stabilizability properties of the nonlinear system through examination of the linearized system.

First we must set up some basic notation. Remember that we are dealing with the linear approximation near $x = 0$. First we take a nonlinear system of the form,

(2.17) $\dot{x} = f(x) + g(x)u$;

where we are interested in finding a continuously differentiable feedback $u = \alpha(x)$ that is defined locally around $x = 0$. We want to show that the corresponding closed loop system,

(2.18) $\dot{x} = f(x) + g(x)\alpha(x)$,

would be locally asymptotically stable at $x = 0$.

Since we are interested in determining to what extent the stabilizability of the above nonlinear system depends on the properties of the linear approximation of the system near $x = 0$, we will expand the system, by Taylor's theorem, where $f_2(x)$ represents all the $f(x)$ terms of order two and greater and $g_1(x)$ represents all the $g(x)$ terms of order one and greater.

(2.19) $$\begin{aligned} f(x) &= Ax + f_2(x) \\ g(x) &= B + g_1(x) \end{aligned}$$

Where $A = \frac{\partial f}{\partial x}(x)\Big|_{x=0}$ and $B = g(0)$

Theorem 2.7 [3, page 173]

Suppose the pair {A, B} from the Jacobian linearization of system (2.19) is (asymptotically) stabilizable. Then any linear feedback which stabilizes the linear approximation will also asymptotically stabilize the original nonlinear system, locally.

If, however the pair {A, B} is not (asymptotically) stabilizable, and has an unstable, uncontrollable eigenspace, that is a λ *with* $Re(\lambda) > 0$ *which "fails" the PBH stabilizability test, then the original nonlinear system is not (asymptotically) stabilizable by any smooth state feedback.*

***Proof**:*

Suppose the linear approximation of the nonlinear system is asymptotically stabilizable. Let F be any matrix such that $(A + BF)$ has all eigenvalues with negative real parts, and set $u = Fx$ on the nonlinear system. The resulting closed loop system is:

(2.20) $\dot{x} = f(x) + g(x)Fx = (A + BF)x + f_2(x) + g_1(x)Fx,$

And its linear approximation is $\dot{x} = (A + BF)x$. This linear approximation has all eigenvalues with negative real parts. Thus, by Theorem 2.3 the nonlinear closed loop system is locally asymptotically stable at $x = 0$.

On the other hand, suppose $\{A, B\}$ is not stabilizable. Then for any feedback matrix F, the matrix $A + BF$ must have at least one eigenvalue with positive real part. Let $u = \alpha(x)$ be any smooth state feedback. Then the corresponding closed loop system has a linear approximation of the form

$$\textbf{(2.21)} \quad \dot{x} = \left[\frac{\partial [f(x) + g(x)\alpha(x)]}{\partial x} \right]_{x=0} x = (A + BF)x \qquad where\ F = \left[\frac{\partial \alpha}{\partial x} \right]_{x=0},$$

which has eigenvalues with positive real part, regardless of the value of α. Note that it is only the linear part of the feedback which can affect the Jacobian linearization of the closed loop system. So in this case, by Theorem 2.3, the nonlinear system is unstable at $x = 0$.

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The case of an eigenvalue with $Re(\lambda) = 0$, which is not stabilizable as determined by a PBH test, is not covered by theorem 2.7. Such cases are known as critical cases of asymptotic stabilization and are not considered in this thesis.

Section 2.6: Detectability

An intuitive concept of detectability is that a system is *detectable* if its unobservable subspace is stable. Recall that the unobservable subspace is the nullspace of W_o.

The precise definition for detectability is:

Definition 2.8:
A system {A, C} is defined to be ***Detectable*** *if there exists a matrix L such that the system* $\dot{x} = (A + LC)x$ *is asymptotically stable.*

This concept is most easily understood through example.

Example 2-8:
We define the following $\{C, A\}$ system as:

(2.22) $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

First we note that the system is not observable.

(2.23) $W_0 = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

The question is; can we create a matrix L such that the system $\dot{x} = (A + LC)x$ is asymptotically stable?

We create the system $\dot{x} = (A + LC)x$.

(2.24) $A + LC = \begin{bmatrix} l_1 + 1 & 0 & 0 \\ l_2 & -2 & 0 \\ l_3 & 0 & -3 \end{bmatrix}$

This system has the following characteristic polynomial:

(2.25) $(\lambda + 2) \cdot (\lambda + 3) \cdot (\lambda - l_1 - 1) = 0$

Note that we need only choose parameter l_1 correctly to have all negative eigenvalues as system (2.22) already has two negative eigenvalues. If we set $l_1 < -1$, then system (2.22) is asymptotically stable. Thus the $\{C, A\}$ is detectable.

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It is interesting to note that the above example provides some nice insight into the intuitive concept of detectability introduced at the beginning of this section. Notice that the observability matrix W_o shows that we can only observe the eigenspace associated with the x_1 component of the system. Similarly it should be noted that l_1 is the only eigenvalue that needs to be controlled in order for the system to be asymptotically stabilized.

What would happen if the only observable component of the system was x_2?

Example 2-9
Utilizing the same *A* matrix as the prior example we examine the effects of a revised C matrix.

(2.26) $C = [0 \quad 1 \quad 0]$

Again we note that the system is not observable.

(2.27) $W_0 = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 4 & 0 \end{bmatrix}$

In this circumstance can we create a matrix *L* such that the system $\dot{x} = (A + LC)x$ is asymptotically stable? Again we create the system $\dot{x} = (A + LC)x$.

(2.28) $$A+LC=\begin{bmatrix}1 & l_1 & 0\\ 0 & l_2-2 & 0\\ 0 & l_3 & -3\end{bmatrix}$$

This system has the following characteristic polynomial:

(2.29) $$(\lambda-1)\cdot(\lambda+3)\cdot(\lambda-l_2+2)=0$$

Note that there is a nonnegative eigenvalue, $\lambda=1$, which we cannot access. Thus *no L* matrix can be devised that will asymptotically stabilize the system $\dot{x}=(A+LC)x$, and as a result the system is *not* detectable.

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In this example we find that by changing the aspects of the system that we observe, the system is no longer detectable. This is because compared to example 2-8 where the unobservable subspace was already stable, in example 2-9 the unobservable subspace is unstable; as evidenced by the non-negative eigenvalue.

It is sometimes difficult to determine if a system is detectable by utilizing definition 2.8. Thus the following PBH Detectability test is provided.

Theorem 2.8 (PBH Detectability Test) [2, page 147]:
The pair {C, A} is detectable if and only if the kernel of

$$\Gamma_\lambda = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$$

is the zero subspace, {0}; for every eigenvalue of A where $RE(\lambda) \geq 0$.

Example 2-10:
Apply the PBH detectability test to the Simple Inverted Pendulum,

(2.30) $$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

In order to see if a system is detectable we calculate the kernel of Γ_λ, for all eigenvalues with $RE(\lambda) \geq 0$. System (2.30) only has one nonnegative eigenvalue $\lambda = 1$, thus:

(2.31) $$\Gamma_\lambda = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}$$

Therefore the kernel of $\Gamma_\lambda = \{0\}$ and we can say that system (2.30) is detectable.

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Section 2.7: Influence of Observable Output on Detectability

To demonstrate the influence the *C* matrix, the observable output, has on Detectability and thus on the Stabilizability of a system we create two sample systems, both utilizing the *A* matrix from examples 2-8 and 2-9, that differ only in their observable output. Note that the *B* matrix is not mentioned; remember that Detectability does not depend on the *B* matrix, only on the matrix pair {*C, A*}.

The following *A* matrix will again be utilized for both examples:

(2.32) $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Remember that this *A* matrix has the following eigenvalues, 1, -2, -3. Thus for the system to be Detectable we need only calculate the Kernel, per Theorem 2.8, for $\lambda = 1$.

Example 2-11:
The first C matrix we examine is defined as:

(2.33) $$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Thus $\Gamma_{\lambda=1}$ equals:

$$\textbf{(2.34)}\quad \Gamma_{\lambda} = \begin{bmatrix} A-\lambda I \\ C \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \\ 1 & 0 & 0 \end{bmatrix}$$

It is readily apparent the dimension of the null-space is 0. Thus we can conclude that with $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ the system $\{C, A\}$ is detectable.

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Example 2-12

The following example utilizes the identical *A* matrix, equation (2.32), as above. However in this example the *C* matrix is changed to:

$$\textbf{(2.35)}\quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

In this case $\Gamma_{\lambda=1}$ equals:

$$\textbf{(2.36)}\quad \Gamma_{\lambda} = \begin{bmatrix} A-\lambda I \\ C \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

It is readily apparent the dimension of the null-space is not 0, it is in fact of dimension 1. Thus we can conclude that with $C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ the system $\{A, C\}$ is *not* detectable. Δ

Chapter 3: Stabilizing the Simple Inverted Pendulum

Section 3.1: Static State Feedback

The most intuitive method of asymptotically stabilizing a system at an equilibrium is to allow the usage of a full 'static' state feedback controller and find a specific case that works.

Example 3-1

In the following example we utilize the closed loop system, $\dot{x} = (A + Bk)x$, from section 2.5, designing a feedback; $k = \begin{bmatrix} -\alpha & -\beta \end{bmatrix}$.

(3.1) $$\dot{x} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -\alpha & -\beta \end{bmatrix} \right) x$$

This can be simplified down by combining terms to become:

(3.2) $$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1-\alpha & -\beta \end{bmatrix} x$$

In order to show that this is stabilizable we must be able to find negative eigenvalues for the system (3.2). Its characteristic polynomial is:

(3.3) $$det(\lambda I - (A + Bk)) = det\begin{pmatrix} \lambda & -1 \\ \alpha - 1 & \lambda + \beta \end{pmatrix} = \lambda^2 + \beta\lambda + \alpha - 1$$

We want $\lambda < 0$; choose a specific eigenvalue, say $\lambda = -1$. In that case we know exactly what form we need to place equation (3.3) into. Since if $\lambda = -1$ then a 2nd degree characteristic polynomial would look like: $(\lambda + 1)^2 = \lambda^2 + 2\lambda + 1$. This means that we can find values $\beta = 2$ and $\alpha = 2$ that allow us to match the characteristic polynomial in (3.3) with the characteristic polynomial from $\lambda = -1$. Thus using those choices for α and β we can say that the inverted pendulum system is stabilizable.

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In a more general fashion we can find the set of values for α and β for which the simple inverted pendulum with full state static feedback is stabilizable.

Example 3-2:
Solving (3.3) for λ we find:

(3.4) $$\lambda = \frac{-\beta}{2} \pm \frac{\sqrt{\beta^2 - 4(\alpha - 1)}}{2}$$

Since we need λ<0; we find that we need β>0 and α>1, in fact we need $\beta > 2\sqrt{\alpha - 1}$. This is true since $\frac{-\beta}{2}$ will dominate $\frac{\sqrt{\beta^2 - 4(\alpha - 1)}}{2}$ as long as $\beta^2 > 4(\alpha - 1)$. Thus we need $\beta > 2\sqrt{\alpha - 1}$. While α >1 ensures that $\sqrt{\alpha - 1}$ is positive, thus ensuring both that $-4(\alpha - 1)$ is negative, as well as β >0.

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Section 3.2: Constant Disturbances

What happens if a system is experiencing a constant disturbance? How would that affect the stabilizability of the system, by static feedback? Imagine the simple inverted pendulum, from section 1.2, mounted in a wind tunnel with a constant wind.

Example 3-3:
If we continue looking at our simple inverted pendulum example about the upper equilibrium, we would see that the new linear system would be:

(3.5) $\ddot{\Phi} - \Phi = u + d$

Where d is a constant disturbance. Therefore the state space representation of the system with a constant disturbance would become:

(3.6) $\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ d \end{bmatrix}$

Looking at the system $u = kx$ where $k = [-\alpha \quad -\beta]$, the system with a disturbance $\dot{x} = (A + Bk)x + d$ becomes:

(3.7) $\dot{x} = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -\alpha & -\beta \end{bmatrix} \right) x + \begin{bmatrix} 0 \\ d \end{bmatrix}$

Combining terms we find:

(3.8) $$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1-\alpha & -\beta \end{bmatrix} x + \begin{bmatrix} 0 \\ d \end{bmatrix}$$

If we split this system back out into two first order differential equations we find that:

(3.9) $$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (1-\alpha)x_1 - \beta x_2 + d \end{aligned}$$

Remember that we defined $x_1 = \Phi$ and $x_2 = \dot{\Phi}$ then:

(3.10) $$\ddot{\Phi} = (1-\alpha)\Phi - \beta\dot{\Phi} + d$$

Writing this equation in standard form we have:

(3.11) $$\ddot{\Phi} + \beta\dot{\Phi} + (\alpha - 1)\Phi = d$$

Note that the general solution of this 2nd order differential equation is, $\Phi_{gen}(t) = \Phi_h(t) + \Phi_p(t)$, where $\Phi_h(t)$ is the solution of the homogenous system and $\Phi_p(t)$ is any particular solution of the non-homogenous system. We know from section 3.1 that $\Phi_h(t) \to 0$ as $t \to \infty$, through the choice of $\beta > 2\sqrt{\alpha - 1}$ and $\alpha > 1$.

To find $\Phi_p(t)$ we utilize the method of undetermined coefficients from linear algebra. Since d is a constant we set $\Phi_p(t) = \text{constant}$. Then substitute our $\Phi_p(t)$ into equation (3.11). We find that the derivative terms become zero and we are left with:

(3.12) $(\alpha - 1)\Phi_p = d$ **so** $\Phi_p = \dfrac{d}{(\alpha - 1)}$

Thus $\Phi_{gen}(t) \longrightarrow \Phi_p = \dfrac{d}{(\alpha - 1)} \neq 0$ as $t \longrightarrow \infty$.

Thus a static linear feedback control can asymptotically stabilize the system at an equilibrium even with a constant disturbance *d*. However the equilibrium that is obtained will not be at $\Phi = 0$ but rather at $\Phi = \dfrac{d}{(\alpha - 1)}$. Thus we would say that Φ has a "steady state error" of Φ_p.

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Section 3.3: Dynamic Feedback

In section 3.2 we demonstrated that a "static" feedback controller which is capable of asymptotically stabilizing a system to the origin may not be capable of asymptotically stabilizing a linear system to the origin, if the system is under the influence of a constant disturbance *d*. This then provides the motivation to look at another class of controllers, *Dynamic feedback controllers*, in the hope that this new form of feedback will prove able to asymptotically stabilize a linear system with disturbance. Dynamic feedback controllers are so called because the feedback utilizes a set of auxiliary variables which involve an integration of state variables.

Definition 3.1
A linear ***dynamic feedback controller*** *is described by the following state space system:*

$$\dot{\xi} = A_c \xi + B_c y$$
$$u = -C_c \xi - D_c y$$

It is important to note that the dynamic feedback controller utilizes *y,* the output of the system to be controlled, as its only input. Subsequently the only output of the dynamic feedback controller is *u*.

Example 3-4
In this example we will utilize the simple inverted pendulum system with a constant disturbance as detailed in equation (3.5).

We will start by adding an auxiliary variable x_0 to the system. This auxiliary variable represents a very simple case of a dynamic feedback controller.

We define this variable to be:

(3.13) $$x_0 = \int_0^t x_1 ds = \int_0^t \Phi(s)ds$$

Now the augmented system will be:

(3.14) $$\begin{aligned} \dot{x}_0 &= x_1 \\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + u + d \end{aligned}$$

The state space representation of the augmented system is:

(3.15) $$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} d$$

As in section 3.1 we create a feedback k such that the closed loop system, $\dot{x} = (A + Bk)x$, is asymptotically stable. Then:

$$\textbf{(3.16)}\quad \dot{x} = \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -\mu & -\alpha & -\beta \end{bmatrix} \right) x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu & 1-\alpha & -\beta \end{bmatrix} x$$

Now we want to find feedback gains α, β, μ such that for all constant disturbance forces $d \in \mathbf{R}$ all solutions of the closed loop system (3.16) converge to the point $(e, 0, 0)$ as $t \longrightarrow \infty$. Thus what we are trying to do is to transfer the steady state error $\Phi = \frac{d}{(\alpha - 1)}$ to a steady state error, *'e'*, in our auxiliary variable x_0.

We can solve the system $\dot{x} = (A + Bk)x$ using the same methodology that we used to solve the static feedback system, in section 3.1. First we determine the characteristic polynomial of $\dot{x} = (A + Bk)x$:

(3.17) $\lambda^3 + \beta\lambda^2 + \lambda(\alpha - 1) + \mu = 0$

We want $\lambda < 0$, choose a specific eigenvalue, say $\lambda = -1$. In that case we know exactly what form we need to place equation (3.17) into. The characteristic polynomial for $\lambda = -1$ is:

(3.18) $(\lambda + 1)^3 = \lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$

Thus we find values $\beta = 3$, $\alpha = 4$ and $\mu = 1$ such that our characteristic polynomial (3.17) matches the characteristic polynomial of $\lambda = -1$. This provides the following closed loop system:

(3.19) $$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\mu & 1-\alpha & -\beta \end{bmatrix} x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} x$$

This system has eigenvalues of -1, -1, -1, as designed.

Thus we can choose values for β, α and μ such that the dynamic closed loop system $\dot{x} = (A + Bk)x$ is asymptotically stable. We have transferred the steady state error from the physical state variables to the auxiliary variables, thus transferring the error from the physical system to a 'virtual' error in the feedback controller.

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Section 3.4: General Structure for Observer based Controller

In order to provide a more powerful method for devising dynamic output feedback, we need to be able to generalize our accomplishments from the last section. The first step to doing this is to take another look at our state space systems:

(3.20)
$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Realize that the only information we have available to us consists of the measured output y, the input u and the known coefficient matrixes A, B and C. The true state of $x(t)$ is not known to us. Since we wish to asymptotically stabilize $x(t)$, we must create a good asymptotic state estimate, $\xi(t)$; where $\xi(t)$ estimates the true state $x(t)$. Thus we make the following definition:

Definition 3.2
$\xi(t)$ *is an* ***Asymptotic Estimate*** *of* x *if* $\lim_{t\to\infty}\left[x(t) - \xi(t)\right] = 0$

We want to be able to find an asymptotic estimate of $x(t)$ using only the output y and the input u, of system (3.20); thus we provide the following definition:

Definition 3.3
An ***Asymptotic State Estimator*** *is a system, with the form* $\dot{\xi} = A_e\xi + B_e y$ *which has the following property: If {u, y} is any input-output pair for (3.20), then the corresponding out put* $\xi(t)$ *of the estimator is an Asymptotic Estimate of the corresponding system state x(t). This is true independent of the initial condition of x(t).*

Thus an asymptotic state estimator for a given system, (3.20), is any system which utilizes $\{u, y\}$ from (3.20) as inputs and has an asymptotic estimate $\xi(t)$, of the state of (3.20) as the output.

The following theorem provides a useful method of determining if an asymptotic state estimator exists for a given system (3.20).

Theorem 3.1 [2, page 145]
System (3.20) is detectable if an asymptotic state estimator exists for the system (3.20).

We now have a methodology to determine if a system is detectable. Once we have identified that a given system is detectable we know, by Theorem 3.1, that we can create an asymptotic state estimator for that system. Thus we now provide a generalized methodology for the creation of an asymptotic state estimator.

We first assume that the matrix pair $\{C, A\}$ is detectable. Then we consider the following system, which is a special case of the general dynamic feedback controller from definition 3.1. We consider the system,

(3.21) $\dot{\xi} = A\xi + Bu - L(y - C\xi)$

to be an asymptotic state estimator for $\{C, A\}$. We call this system (3.21) an ***observer system***.

Note that the matrix L is the only undefined term in the above system; L is chosen so that $\xi(t)$ asymptotically estimates $x(t)$. Our aim is to define a matrix L such that we achieve an estimation error that goes to zero as t approaches infinity.

We define our error to be:

(3.22) $e := x - \xi$

We show through algebraic manipulation that:

(3.23)
$$\begin{aligned}
\dot{e} &= \dot{x} - \dot{\xi} \\
\dot{e} &= Ax + Bu - A\xi - Bu + L(y - C\xi) \\
\dot{e} &= Ae + L(y - C\xi) \\
\dot{e} &= Ae + L(Cx - C\xi) \\
\dot{e} &= Ae + LCe \\
\dot{e} &= (A + LC)e
\end{aligned}$$

This system should seem familiar from the discussion of Detectability in Chapter 2. We can achieve $\lim_{t\to\infty} e = 0$ by designing L appropriately. Since we assumed that the system $\{C, A\}$

was detectable to start with, then by definition 2.8 we can define an *L* such that the system (*A*+*LC*) is asymptotically stable to *x(t)*.

The question then becomes: How do we devise such an *L*?

First we define *u* to be:

(3.24) $u = K\xi$

The matrix *K* is devised so that the asymptotic estimator $\xi(t)$, is asymptotically stable to the origin. The matrix *L* and the matrix *K* must both be designed correctly so that the observer system is asymptotically stable to the origin, thus ensuring that *x(t)* is also asymptotically stable to the origin.

We can use (3.20) along with (3.21) and (3.24) to write the combined system for (x, ξ) as:

(3.25) $\begin{bmatrix} \dot{x} \\ \dot{\xi} \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & A+LC+BK \end{bmatrix} \begin{bmatrix} x \\ \xi \end{bmatrix}$

A more transparent structure is available though displaying the system is terms of (x, e). In this set of variables the closed loop system is:

(3.26) $$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A+BK & -BK \\ 0 & A+LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

The beautiful aspect of showing the system in this form is that it is an upper block triangular matrix. Thus it becomes a simpler matter to choose the *K* and *L* matrixes such that the system has eigenvalues with a negative real part, since the two matrixes are now decoupled. In this fashion the feedback control, the matrix *K*, and the matrix *L* can be designed independently of each other so that the estimated system, $\xi(t)$, asymptotically goes to *x(t),* while *x(t)* is asymptotically stabilized.

Theorem 3.2 [2]
There exists an observer based dynamic feedback controller for: $\dot{x} = Ax + Bu, \quad y = Cx$

If and only if the pair {A, B} is stabilizable and the pair {C, A} is detectable.

This makes good intuitive sense. We have already established that the *K* and *L* matrices are an integral part of the observer based dynamic system. Having appropriate *K* and *L* matrices which provide stabilizability and detectability is therefore obviously required for an observer based dynamic controller. It is slightly less obvious that the reverse is also true, if an observer based dynamic controller can be created then system (3.26) is the only form you need examine. Other forms of observer based dynamic controllers may work but (3.26) is guaranteed. As such (3.26) guarantees that the pair {*A, B*} is stabilizable and the pair {*C, A*} is detectable

Section 3.5: Dynamic Output Feedback of the Simple Inverted Pendulum

We can utilize our Simple Inverted Pendulum system to demonstrate the concepts from section 3.4.

Example 3-5:
In this first example only the position measurement is observable.

(3.27) $$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

The first step is to realize that the system is detectable; we showed this in example 2-10. We also know it to be true since we already demonstrated that the system is observable and observability implies detectability.

Thus we can form the system as shown in equation (3.26) in order to find values for *L* and *K*.

(3.28) $$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ K_1 + 1 & K_2 & -K_1 & -K_2 \\ \hdashline 0 & 0 & L_1 & 1 \\ 0 & 0 & L_2 + 1 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ e_1 \\ e_2 \end{bmatrix}$$

It is easy to choose values for *K* and *L* so as to produce eigenvalues with negative real parts. For example choose terms of *L* such that the lower right block has eigenvalues -1, -1 and choose terms for *K* such that the upper left block has eigenvalues -1, -1. Then it is easy to see that we much choose K_1=-2, K_2=-2, L_1=-2 and L_2=-2. Therefore the system has the closed loop form:

(3.29) $$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ -1 & -2 & 2 & 2 \\ \hdashline 0 & 0 & -2 & 1 \\ 0 & 0 & -1 & 0 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ e_1 \\ e_2 \end{bmatrix}$$

As expected system (3.29) has eigenvalues -1,-1,-1,-1. Thus the choices of *K* and *L* guarantee that the combined system is asymptotically stabilizable using dynamic feedback.

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Example 3-6
What happens if we change the *C* matrix? Is the system still detectable? Still stabilizable?

Check with C matrix:

(3.30) $$\mathrm{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \mathrm{x}$$

Our intuition tells us that the system is still detectable, and thus stabilizable. We already know, from example 2-3, that while observing only x_2 the system is observable and observability is stronger then detectability. Thus we can form the system (3.26) in order to find values for *L* and *K*, which will asymptotically stabilize the system with this *C*.

(3.31) $$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ K_1+1 & K_2 & -K_1 & -K_2 \\ \hdashline 0 & 0 & 0 & L_1+1 \\ 0 & 0 & 1 & L_2 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ e_1 \\ e_2 \end{bmatrix}$$

We can choose values for *K* and *L*, as we did in the above example, so as to produce eigenvalues with negative real parts. This time we choose terms of *L* such that the lower right block has eigenvalues -2, -2 and choose terms for *K* such that the upper left block has eigenvalues -3, -3. Then it is easy to see that we much choose K_1=-10, K_2=-6, L_1=-5 and L_2=-4. Therefore the system has the closed loop form:

(3.32) $$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \left[\begin{array}{cc:cc} 0 & 1 & 0 & 0 \\ -9 & -6 & 10 & 6 \\ \hdashline 0 & 0 & 0 & -4 \\ 0 & 0 & 1 & -4 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ e_1 \\ e_2 \end{bmatrix}$$

As expected system (3.32) has eigenvalues -3,-3,-2,-2. Thus the choices of *K* and *L* guarantee that the combined system is asymptotically stabilizable using dynamic feedback.

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Chapter 4: Stabilizing the Pendulum on a Cart (PoC)

Section 4.1: PBH Test for Linear Stabilizability

Prior to examining the specifics regarding stabilization of the Pendulum on a Cart system utilizing either Static State or Observer Based Dynamic feedback it is important to remember that we have not yet even shown that the Pendulum on a Cart system is stabilizable by any linear feedback. In order to prove this simple yet vitally important fact we will use the PBH test for Linear Stabilizability from Chapter Two.

First we will display the A and B matrices we derived for the Pendulum on a Cart system, back in section 1.3, at the beginning of this chapter for ease of reference.

(4.1) $$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{Ml} & 0 \end{bmatrix}$$

(4.2) $$B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{-1}{Ml} \end{bmatrix}$$

Note that we can choose our units such that $m = 1$, $g = 1$ and $l = 1$; however since M >> m, we cannot choose M =1; for the moment it will be advantageous to us to leave M as an arbitrarily large finite set value. Once these unit choices are made then equations 4.1 and 4.2 become:

(4.3) $$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-1}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{M+1}{M} & 0 \end{bmatrix}$$

(4.4) $$B = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{-1}{M} \end{bmatrix}$$

Next we use Theorem 2.6, the PBH test for linear stabilizability, to determine if the Pendulum on a Cart system is stabilizable using linear feedback. Remember that the test requires that $\text{Rank}[A - \lambda I \quad B] = n$ hold true for every $Re(\lambda) \geq 0$ eigenvalue. For the Pendulum on a Cart system we have the following eigenvalues: 0, 0, $\frac{\sqrt{M(M+1)}}{M}$ and -$\frac{\sqrt{M(M+1)}}{M}$. Thus we must check to ensure that the rank test holds for both the zero and the positive eigenvalue.

For $\lambda = \frac{\sqrt{M(M+1)}}{M}$:

(4.5) $$[A - \lambda I \quad B] = \begin{bmatrix} -\frac{\sqrt{M(M+1)}}{M} & 1 & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{M(M+1)}}{M} & \frac{-1}{M} & 0 & \frac{1}{M} \\ 0 & 0 & -\frac{\sqrt{M(M+1)}}{M} & 1 & 0 \\ 0 & 0 & \frac{M+1}{M} & -\frac{\sqrt{M(M+1)}}{M} & \frac{-1}{M} \end{bmatrix}$$

This obviously has a rank 4.

For $\lambda = 0$:

(4.6) $$[A-\lambda I \quad B]=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{-1}{M} & 0 & \frac{1}{M} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{(M+1)}{M} & 0 & \frac{-1}{M} \end{bmatrix}$$

This also has a rank 4.

Thus since for all non-negative eigenvalues the system $[A-\lambda I \quad B]$ has rank = n, then as expected the Pendulum on a Cart system is stabilizable by some linear feedback. It is important to note that this property; that of the stabilizability of the system by linear feedback, is not related in any fashion to the value of M; as long of course as M >> m.

Section 4.2: Static State Feedback of the Pendulum on a Cart

As in the case of the simple inverted pendulum, the logical first step in dealing with the stability of the pendulum on a cart is to examine if the system can be stabilized using static state feedback and if so, what some of the system parameters are for such a stabilization. As with the case of the simple inverted pendulum we set $u = kx$ such that $\dot{x} = (A + Bk)x$. However since the Pendulum on a Cart system has four system variables, we will define the *k* matrix as:

(4.7) $$k = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}$$

Thus $\dot{x} = (A + Bk)x$ becomes:

(4.8) $$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{k_1}{M} & \frac{k_2}{M} & \frac{-1}{M} + \frac{k_3}{M} & \frac{k_4}{M} \\ 0 & 0 & 0 & 1 \\ \frac{-k_1}{M} & \frac{-k_2}{M} & \frac{(M+1)}{M} - \frac{k_3}{M} & \frac{-k_4}{M} \end{bmatrix} x$$

The characteristic polynomial of this matrix is then:

(4.9) $\lambda^4 + \lambda^3\left(\frac{k_4}{M} - \frac{k_2}{M}\right) + \lambda^2\left(-\frac{k_1}{M} + \frac{k_3}{M} - 1 - \frac{1}{M}\right) + \lambda\left(\frac{k_2}{M}\right) + \frac{k_1}{M}$

We choose a negative eigenvalue, say $\lambda = -1$. We know the characteristic polynomial for $\lambda = -1$ is:

(4.10) $(\lambda + 1)^4 = \lambda^4 + 4\lambda^3 + 6\lambda^2 + 4\lambda + 1$

Thus we can solve for the values of k_i, such that the characteristic polynomial for our system, the pendulum on a cart, is the same as the characteristic polynomial for $\lambda = -1$, thus guaranteeing that our system has an eigenvalue of -1, with a geometric multiplicity of 4, and is thus asymptotically stable.

The values of k_i for which this works are:

k_1 = M

k_2 = 4M

k_3 = 8M + 1

k_4 = 8M

It is worth noting again that the value of M is not relevant to the fundamental discussion regarding the k_i parameters for which the system is stable. As long as M is a finite positive number, where M >> m then the system dynamics are not affected.

In Section 3.3 we found that it was feasible to find the set of all k_i values for which the simple inverted pendulum system was stable. For the pendulum on cart system this type of solution is no longer feasible. It would require solving a 4th order equation, equation 4.9, for the k_i values; in general there are no explicit solutions to higher order equations.

Section 4.3: PBH Detectability Test

Prior to determining some parameters for an Observer Based Feedback Controller of the Pendulum on a Cart system we need to ensure that a good estimator exists for the system. That is we need to determine if an asymptotic state estimator exists, to do this we will use the PBH detectability test from Theorem 2.8.

Since the PBH detectability test utilizes a *C* matrix, we need to define a particular *C* matrix for the system to determine if the $\{C, A\}$ system is detectable for that given *C* matrix.

Example 4-1
We shall define the C matrix to be:

(4.11) $C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$

Remember that to use the PBH detectability test we need to know the eigenvalues for the *A* matrix. We already computed these values back in section 4.1. So we will directly apply them here.

For $\lambda = \frac{\sqrt{M(M+1)}}{M}$:

$$\textbf{(4.12)}\quad \Gamma_\lambda = \begin{bmatrix} A-\lambda I \\ \\ C \end{bmatrix} = \begin{bmatrix} \frac{-\sqrt{M(M+1)}}{M} & 1 & 0 & 0 \\ 0 & \frac{-\sqrt{M(M+1)}}{M} & \frac{-1}{M} & 0 \\ 0 & 0 & \frac{-\sqrt{M(M+1)}}{M} & 1 \\ 0 & 0 & \frac{(M+1)}{M} & \frac{-\sqrt{M(M+1)}}{M} \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

We then determine what the kernel of the Γ_λ matrix is. The kernel is the zero subspace.

For $\lambda = 0$:

$$\textbf{(4.13)}\quad \Gamma_\lambda = \begin{bmatrix} A-\lambda I \\ \\ C \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-1}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+1)}{M} & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The kernel of this matrix is also the zero subspace.

Since both the non-negative eigenvalues of the A matrix have the zero subspace for the Γ_λ matrix we say that the given $\{A, C\}$ system is detectable. Thus an asymptotic state estimator can be found for the system and we can create an Observer Based Feedback controller for the system.

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Our fortuitous choice for the C matrix provides us with a vitally important piece of information. Notice that when dealing with $\lambda = 0$ the only entry in the 1^{st} column of the Γ_λ matrix, was from the C matrix. If the C matrix had been designed in any fashion such that this entry, which corresponds to the x_1 variable, was to be left blank, then the kernel of the Γ_λ matrix would not be the zero subspace. The column rank of the Γ_λ matrix would be less then n, providing a kernel of [1, 0, 0, 0]. *Thus no asymptotic state estimator could exist for the system and the system would not be stabilizable by Observer Based Dynamic Feedback.*

This idea is easily demonstrated looking at a further example.

Example 4-2:

Take the C matrix, $C = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix}$. We expect, based on our prior observations, that since the c_1 entry is zero, the resulting system will prove not to be detectable.

We start off the same as before, utilizing the eigenvalues for the A matrix.

For $\lambda = \frac{\sqrt{M(M+1)}}{M}$:

$$\textbf{(4.14)} \quad \Gamma_\lambda = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = \begin{bmatrix} \frac{-\sqrt{M(M+1)}}{M} & 1 & 0 & 0 \\ 0 & \frac{-\sqrt{M(M+1)}}{M} & \frac{-1}{M} & 0 \\ 0 & 0 & \frac{-\sqrt{M(M+1)}}{M} & 1 \\ 0 & 0 & \frac{(M+1)}{M} & \frac{-\sqrt{M(M+1)}}{M} \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Determine the kernel of the Γ_λ matrix. The kernel is the zero subspace.

For $\lambda = 0$:

(4.15) $$\Gamma_\lambda = \begin{bmatrix} A-\lambda I \\ \\ C \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-1}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+1)}{M} & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

Providing a dimension of 1 with a kernel, as expected, of [1, 0, 0, 0].

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Section 4.4: Observer Based Dynamic Feedback Controller

Now that we have determined that the Pendulum on a Cart system, system {*C, A*}, is detectable via theorem 2.8, we can create an Observer Based Dynamic Feedback Controller. Remember however that this has only been accomplished in the previous section for the given *C* matrix $C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$. We can now utilize the following system, derived in Section 3.4, in order to find *K* and *L* matrices which will asymptotically stabilize the Pendulum on a Cart system:

(4.16) $$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A + BK & -BK \\ 0 & A + LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

As before we will define the *K* matrix as:

(4.17) $$K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}$$

We shall define the *L* matrix as:

(4.18) $$L = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix}$$

Thus the system, (4.16), becomes:

$$
\textbf{(4.19)}\quad
\left[\begin{array}{ccc:ccccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_1/M & k_2/M & -1/M + k_3/M & k_4/M & -k_1/M & -k_2/M & -k_3/M & -k_4/M \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-k_1/M & -k_2/M & (M+1)/M - k_3/M & -k_4/M & k_1/M & k_2/M & k_3/M & k_4/M \\
\hdashline
0 & 0 & 0 & 0 & l_1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & l_2 & 0 & -1/M & 0 \\
0 & 0 & 0 & 0 & l_3 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & l_4 & 0 & (M+1)/M & 0
\end{array}\right]
$$

As mentioned previously the beautiful thing about the form of this matrix is that as an upper block triangular matrix it is possible to solve for the *L* and *K* matrices separately. As can be seen this is a very important attribute particularly as the dimension of the problem climbs. While solving the Simple Inverted Pendulum, a 4x4 system, may not be so bad, solving an 8x8 system like the Pendulum on a Cart would be very tedious. The ability to decouple the calculations and solve for *K* and *L* separately greatly reduces the difficulty of the task.

It is also important to note that the upper left block of this matrix is exactly the same as the matrix from equation 4.8. That is, solving to find a *K* matrix that statically stabilizes a system is identical to finding a *K* matrix that dynamically stabilizes the system utilizing observer based feedback. This makes good intuitive sense. The *K* matrix is always responsible for ensuring that the system in question is asymptotically stable to the origin; in the case of static feedback the system that is being asymptotically stabilized is the *x(t)* system; while in the case of observer based dynamic feedback the *K* matrix is ensuring that the asymptotic estimator is being asymptotically stabilized to the origin, thereby ensuring that the *x(t)* system is also asymptotically stabilized to the origin.

The practical upshot of this observation is that we can utilize the K matrix we derived in the previous section, as it will still provide for negative eigenvalues of -1, -1, -1 and -1. Thus we already know that $K = [M \quad 4M \quad 8M+1 \quad 8M]$. This leaves us with only having to find values for the *L* matrix that will provide negative eigenvalues. To accomplish this we will use the familiar pattern of choosing an eigenvalue, $\lambda = -1$, finding the characteristic polynomial for that eigenvalue:

(4.20) $$(\lambda+1)^4 = \lambda^4 + 4\lambda^3 + 6\lambda^2 + 4\lambda + 1$$

then solving for values of l_i such that the characteristic polynomial of the lower right block of matrix 4.19 has an identical form.

The polynomial of the lower right block of matrix 4.19 is:

(4.21) $$\lambda^4 - \lambda^3 l_1 + \lambda^2\left(-l_2 - 1 - \frac{1}{M}\right) + \lambda\left(l_1 + \frac{l_1}{M} + \frac{l_3}{M}\right) + l_2 + \frac{l_2}{M} + \frac{l_4}{M}$$

Thus

(4.22) $$L = \begin{bmatrix} -4 \\ \dfrac{(7M+1)}{M} \\ 8M+4 \\ \dfrac{8M^2+(8M+1)}{M} \end{bmatrix}$$

Then the resulting system is:

(4.23)

$$\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 4 & \dfrac{-1}{M}+\dfrac{(8M+1)}{M} & \dfrac{8}{M} & -1 & -4 & \dfrac{-(8M+1)}{M} & \dfrac{-8}{M} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
-1 & -4 & \dfrac{(M+1)}{M}-\dfrac{(8M+1)}{M} & \dfrac{-8}{M} & 1 & 4 & \dfrac{(8M+1)}{M} & \dfrac{8}{M} \\
0 & 0 & 0 & 0 & -4 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -\dfrac{(7M+1)}{M} & 0 & \dfrac{-1}{M} & 0 \\
0 & 0 & 0 & 0 & 8M+4 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 8M^2+\dfrac{(8M+1)}{M} & 0 & \dfrac{(M+1)}{M} & 0
\end{bmatrix}$$

By construction this system has an eigenvalue of -1 with a geometric multiplicity of 8. Thus the choices of *K* and *L* guarantee that the combined system is asymptotically stable using dynamic observer based feedback. In a practical application the *L* matrix would be designed with eigenvalues of a greater magnitude then the *K* matrix. This would allow for a "better" asymptotic estimate to be used in the state feedback, by providing the *K* matrix with as much help as possible.

Section 4.5: Static Feedback of the PoC with Constant Disturbance

What happens to the Pendulum on a Cart system if it is experiencing a constant disturbance? Imagine the disturbance is entering the system through the x_2 variable, the velocity of the pendulum.

The formula for the Pendulum on a Cart with a disturbance entering through x_2 is:

$$\textbf{(4.24)}\quad \dot{x} = (A + Bk)x + \begin{bmatrix} 0 \\ d \\ 0 \\ 0 \end{bmatrix}$$

Remember we have already defined the A and B matrixes, carrying out the matrix algebra we find that:

$$\textbf{(4.25)}\quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{k_1}{M} & \frac{k_2}{M} & -\frac{1}{M} + \frac{k_3}{M} & \frac{k_4}{M} \\ 0 & 0 & 0 & 1 \\ -\frac{k_1}{M} & -\frac{k_2}{M} & \frac{M+1}{M} - \frac{k_3}{M} & -\frac{k_4}{M} \end{bmatrix} x = \begin{bmatrix} 0 \\ -d \\ 0 \\ 0 \end{bmatrix}$$

Thus it is immediate that $x_2 = 0$ and $x_4 = 0$.

This leaves us with two equations and two unknowns, which we can solve for. We solve:

(4.26)
$$\frac{k_1}{M}x_1 + \left(-\frac{1}{M} + \frac{k_3}{M}\right)x_3 = -d$$
$$-\frac{k_1}{M}x_1 + \left(\frac{M+1}{M} - \frac{k_3}{M}\right)x_3 = 0$$

If we add the two equations together we can solve for x_3, and we find that $x_3 = -d$

Then inserting x_3 back into the second equation we get:

(4.27) $$x_1 = \frac{(m+1-k_3)d}{-k_1}$$

We assigned values for k_1 and k_3 in Section 4.2, utilizing the same k values that worked to statically stabilize the system without a disturbance is a logical step, inserting those values into the above equation we find that $x_1 = 7d$

This simple set of calculation confirms our initial suspicions. The Pendulum on a Cart system with static feedback will stabilize itself under a constant disturbance but the system will not reach a zero equilibrium, rather it will reach a steady state at:

(4.28) $$x = \begin{bmatrix} 7d \\ 0 \\ -d \\ 0 \end{bmatrix}$$

It is interesting to note that the error which entered the system through the pendulums velocity is being exhibited as an error in both the pendulum's and the carts position.

Section 4.6: Dynamic Stabilization of the PoC with Constant Disturbance

We know that we can stabilize the Pendulum on a Cart by utilizing Observer Based Dynamic Feedback. This was proven in Section 4.4. Since the $\{A, B\}$ pair is stabilizable, and the $\{C, A\}$ pair is detectable, the Pendulum on a Cart system can be stabilized with Observer Based Dynamic Feedback, even with a constant disturbance.

However, is it possible to stabilize the system with a constant disturbance without resorting to the full 8x8 system as laid out in that section?

The most obvious starting place is to note that under the influence of a constant disturbance the steady state error is found in the x_1 and x_3 variables. Thus it is logical to try introducing only two auxiliary variables; one for each x_1 and x_3. The concept of these auxiliary variables is to transpose the physical steady state error from the state variables into the auxiliary variables; thus displacing the physical error into our control feedback mechanism. Since the steady state error is seen in the x_1 and x_3 variables we introduce two associated auxiliary variables z_1 and z_3. Where:

(4.29)
$$z_1 = \int_0^t x_1 ds$$
$$z_3 = \int_0^t x_3 ds$$

Thus the augmented system would look like:

$$\begin{aligned}
\dot{z}_1 &= x_1 \\
\dot{z}_3 &= x_3 \\
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{-1}{M}x_3 + \frac{u}{M} \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{(M+1)}{M}x_3 - \frac{u}{M}
\end{aligned} \quad \textbf{(4.30)}$$

The state space representation of this system is then:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{M} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{(M+1)}{M} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{M} \\ 0 \\ \frac{-1}{M} \end{bmatrix} u \quad \textbf{(4.31)}$$

Next we use Theorem 2.6, the PBH test for linear stabilizability, to determine if the Pendulum on a Cart system with this constant disturbance is stabilizable using linear feedback. Remember that the test requires that $\text{Rank}[A-\lambda I \quad B]=n$ hold true for every $Re(\lambda)\geq 0$ eigenvalue. For the above augmented system we have the following eigenvalues: 0, 0, 0, 0, $\frac{\sqrt{M(M+1)}}{M}$ and $-\frac{\sqrt{M(M+1)}}{M}$. Thus we must check to ensure that the rank test holds for both the zero and the positive eigenvalue.

For $\lambda = 0$:

(4.32) $$[A-\lambda I \quad B]=\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{M} & 0 & \frac{1}{M} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{(M+1)}{M} & 0 & \frac{-1}{M} \end{bmatrix}$$

This has a rank of 5. The 6th row is a linear combination of the 2nd and 4th rows. Since the rank is 5 and the rank of the augmented *A* matrix is 6, then according to Theorem 2.6, the system is not stabilizable utilizing linear feedback.

Thus we can not stabilize the system with the reduced dynamic feedback controller we have created. This is not to say that all reduced size dynamic controllers will not work, it is conceivable that some other configuration of auxiliary variables will pass the PBH test for linear stabilizability.

Chapter 5: Further Study

The most immediate question left unanswered is if it is possible to stabilize the Inverted Pendulum on a Cart system, with a constant disturbance, without resorting to the full 8 x 8 system. Since the choices for auxiliary variables, which were utilized in section 4.6, were the most logical initial choices further investigations into the problem are required.

Other questions of interest and practical importance deal with the optimality of the feedback controller. How can the system be defined so as to minimize the cost of the controller? Depending on the application of the controller various methodologies may be employed, for instance by minimizing number of observed state space variables the cost may be reduced. It may also be possible that there are different costs associated with observing different variables, in which case a linear optimization may be needed to determine which set of variables would minimize the overall cost. On the other hand minimizing the cost of the controller may merely be a function of minimizing the time it takes to asymptotically stabilize the system, or even minimizing the time during which the system is out of some predefined ‘acceptable’ range around the equilibrium. These ideas are examined in linear optimal control theory.

Another topic of interest is determining the “basin of attraction”, *D*, around the equilibrium for which the Jacobian linearization determines the actions of the estimates the nonlinear system. The basin of attraction is the set of initial conditions for the nonlinear system that lead to trajectories that asymptotically converge to the equilibrium. This idea is examined in Lyapanov stability theory for nonlinear systems.

All of these topics can be studied utilizing the examples of the Simple Inverted Pendulum and the Inverted Pendulum on a Cart. Analysis of these topics is left unanswered in this paper and are put forth to invite further study on the topic.

Appendix A: Summary of Theorems

Theorem 2.1 [2, page 43]:
A system is observable if and only if Wo has full column rank

Theorem 2.2 [2, page 20]:
A system, $\dot{x} = Ax$ *is asymptotically stable if and only if all the eigenvalues,* λ_i*, of A have negative real part, that is* $Re(\lambda_i)$ *are negative.*

Theorem 2.3 [4, page 139]:
Let x = 0 be an equilibrium point for the nonlinear system:

$$\dot{x} = f(x)$$

Where f: D $\longrightarrow R^n$ *is continuously differentiable and D is a neighborhood of the origin. Let*

$$A = \frac{\partial f}{\partial x}(x)\bigg|_{x=0}$$

Then,

3. *The origin is asymptotically stable if* $\mathrm{Re}(\lambda_i) < 0$ *for all eigenvalues,* λ_i*, of A.*
4. *The origin is unstable if* $\mathrm{Re}(\lambda_i) > 0$ *for one or more eigenvalues,* λ_i*, of A.*

Theorem 2.4 [2, page 57]
The Kalman controllability rank test says that the system $\dot{x} = Ax + Bu$ *is controllable if and only if rank* $W_c := rank[B \;\; AB \;\; A^2B \;\; \ldots \;\; A^{n-1}B] = n$*, where A is an n x n matrix.*

Theorem 2.5 [2, page 59]:
The PBH controllability test says that the system $\dot{x} = Ax + Bu$ is controllable, that is the matrix pair {A, B} is controllable, if and only if $rank[A - \lambda I \quad B] = n$ for all complex eigenvalues λ of A.

Theorem 2.6 [2, page 123]
The PBH test for linear stabilizability says that the pair {A, B} is stabilizable if and only if $Rank[A - \lambda I \quad B] = n$ for every eigenvalue λ of A with non-negative real part (and thus for every complex λ in the right half plane).

Theorem 2.7 [3, page 173]
Suppose the pair {A, B} from the Jacobian linearization of the system

$$f(x) = Ax + f_2(x)$$
$$g(x) = B + g_1(x)$$

is (asymptotically) stabilizable. Then any linear feedback which stabilizes the linear approximation will also asymptotically stabilize the original nonlinear system, locally. If, however the pair {A, B} is not (asymptotically) stabilizable, and has an unstable, uncontrollable eigenspace, that is a λ with $Re(\lambda) > 0$ which "fails" the PBH stabilizability test, then the original nonlinear system is not (asymptotically) stabilizable by any smooth state feedback.

Theorem 2.8 (PBH Detectability Test) [2, page 147]:
The pair {C, A} is detectable if and only if the kernel of

$$\Gamma_\lambda = \begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$$

is the zero subspace, {0}; for every eigenvalue of A where $RE(\lambda) \geq 0$.

Theorem 3.1 [2, page 145]
The system:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

is detectable if an asymptotic state estimator exists for the system.

Theorem 3.2 [2]
There exists an observer based dynamic feedback controller for: $\dot{x} = Ax + Bu, \quad y = Cx$

If and only if the pair {A, B} is stabilizable and the pair {C, A} is detectable.

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